DIRECT AND INVERSE RESULTS ON ROW SEQUENCES OF HERMITE-PADÉ APPROXIMANTS

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Dedicated to A.A. Gonchar, on the occasion of his eightieth birthday

Abstract. We give necessary and sufficient conditions for the convergence with geometric rate of the common denominators of simultaneous rational interpolants with a bounded number of poles. The conditions are expressed in terms of intrinsic properties of the system of functions used to build the approximants. Exact rates of convergence for these denominators and the simultaneous rational approximants are provided.

1. Introduction

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of d formal or convergent Taylor expansions about the origin; that is, for each $k = 1, \ldots, d$, we have

$$f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} z^n, \qquad \phi_{n,k} \in \mathbb{C}. \tag{1}$$

Let $\mathbf{D} = (D_1, \ldots, D_d)$ be a system of domains such that, for each $k = 1, \ldots, d$, f_k is meromorphic in D_k. We say that the point ξ is a pole of $\mathbf{f}$ in $\mathbf{D}$ of order τ if there exists an index $k \in \{1, \ldots, d\}$ such that $\xi \in D_k$ and it is a pole of f_k of order τ, and for $j \neq k$ either ξ is a pole of f_j of order less than or equal to τ or $\xi \notin D_j$. When $\mathbf{D} = (D, \ldots, D)$ we say that ξ is a pole of $\mathbf{f}$ in D.

Let $R_0(\mathbf{f})$ be the largest disk in which all the expansions $f_k, k = 1, \ldots, d$ correspond to analytic functions. If $R_0(\mathbf{f}) = 0$, we take $D_m(\mathbf{f}) = \emptyset, m \in \mathbb{Z}_+$; otherwise, $R_m(\mathbf{f})$ is the radius of the largest disk $D_m(\mathbf{f})$ centered at the origin to which all the analytic elements $(f_k, D_0(f_k))$ can be extended so that $\mathbf{f}$ has at most m poles counting multiplicities. The disk $D_m(\mathbf{f})$ constitutes for systems of functions the analogue of the m-th disk of meromorphy defined

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by J. Hadamard in [5] for $d = 1$. Moreover, in that case both definitions coincide.

By $\mathcal{Q}_m(\mathbf{f})$ we denote the monic polynomial whose zeros are the poles of $\mathbf{f}$ in $D_m(\mathbf{f})$ counting multiplicities. The set of distinct zeros of $\mathcal{Q}_m(\mathbf{f})$ is denoted by $\mathcal{P}_m(\mathbf{f})$.

Definition 1.1. *Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of d formal Taylor expansions as in* (1)*. Fix a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the zero vector in $\mathbb{Z}_+^d$. Set $|\mathbf{m}| = m_1 + \dots + m_d$. Then, for each $n \geq \max\{m_1, \dots, m_d\}$, there exist polynomials $Q, P_k, k = 1, \dots, d$, such that*

a.1) $\deg P_k \leq n - m_k$, $k = 1, \dots, d$, $\quad \deg Q \leq |\mathbf{m}|$, $\quad Q \not\equiv 0$,
a.2) $Q(z)f_k(z) - P_k(z) = A_k z^{n+1} + \cdots$.

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \dots, P_d/Q)$ is called an $(n, \mathbf{m})$ Hermite-Padé approximation of $\mathbf{f}$.

This vector rational approximation, in general, is not uniquely determined and in the sequel we assume that given $(n, \mathbf{m})$ one particular solution is taken. For that solution we write

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \dots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}, \tag{2}$$

where $Q_{n,\mathbf{m}}$ has no common zero simultaneously with all the $P_{n,\mathbf{m},k}$ and is normalized to be monic unless otherwise stated. Sequences $\{\mathbf{R}_{n,\mathbf{m}}\}$ for which $|\mathbf{m}|$ remains fixed when n varies are called row sequences, and when $|\mathbf{m}| = \mathcal{O}(n)$, $n \to \infty$, diagonal sequences.

The study of simultaneous Hermite-Padé approximations of systems of functions has a long tradition (see [6]) and they have been subject to renewed interest in the recent past (see, for instance, [3] and the references therein). Many papers deal with diagonal sequences and their applications in different fields (number theory, random matrices, brownian motions, Toda lattices, to name a few). At the same time, few papers study row sequences. In this second direction a significant contribution is due to Graves-Morris/Saff in [8] where they prove an analogue of the Montessus de Ballore theorem which plays a central role in the classical theory of Padé approximation. See also [9]-[10] for different approaches to the same type of results as well as [11] and references therein for least-squares versions.

Before going into details let us briefly describe the scalar case ($d = 1$) corresponding to classical Padé approximation which is well understood. When $d = 1$ we write $\mathbf{f} = f$, $\mathbf{m} = m \in \mathbb{N}$, and $\mathbf{R}_{n,\mathbf{m}} = R_{n,m}$. Given a compact set $K \subset \mathbb{C}$, $\|\cdot\|_K$ denotes the sup norm on K. We summarize what we need in the following statement.

Gonchar's Theorem. *Let f be a formal Taylor expansion about the origin and fix $m \in \mathbb{N}$. Then, the following two assertions are equivalent.*

a) *$R_0(f) > 0$ and f has exactly m poles in $D_m(f)$ counting multiplicities.*

b) *There is a polynomial* Q_m *of degree* m, $Q_m(0) \neq 0$, *such that the sequence of denominators* $\{Q_{n,m}\}_{n \geq m}$ *of the Padé approximations of* f *satisfies*

$$\limsup_{n\to\infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ *denotes the coefficient norm in the space of polynomials.*

Moreover, if either a) or b) takes place then $Q_m \equiv \mathcal{Q}_m(f)$,

$$\theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(f)\}}{R_m(f)}, \tag{3}$$

and

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)}, \tag{4}$$

where K *is any compact subset of* $D_m(f) \setminus \mathcal{P}_m(f)$.

From this result it follows that if ξ is a pole of f in $D_m(f)$ of order τ, then for each $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, $Q_{n,m}$ has exactly τ zeros in $\{z : |z - \xi| < \varepsilon\}$. We say that each pole of f in $D_m(f)$ attracts as many zeros of $Q_{n,m}$ as its order when n tends to infinity.

So stated Gonchar's Theorem does not appear in the literature and needs some comments. Under assumptions a), in [7] Montessus de Ballore proved that

$$\lim_{n\to\infty} Q_{n,m} = \mathcal{Q}_m(f), \qquad \lim_{n\to\infty} R_{n,m} = f,$$

with uniform convergence on compact subsets of $D_m(f)\setminus\mathcal{P}_m(f)$ in the second limit. In essence, Montessus proved that a) implies b) with $Q_m = \mathcal{Q}_m(f)$, showed that $\theta \leq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and proved (4) with equality replaced by $\leq$. These are the so called direct statements of the theorem. The inverse statements, b) implies a), $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and the inequality $\geq$ in (4) are immediate consequences of [4, Theorem 1]. The study of inverse problems of Padé approximation was suggested by A.A. Gonchar in [4, Subsection 12] where he presented some interesting conjectures. Some of them were solved in [12] and [13]. See [1] for a brief account of Gonchar's most recent results and a list of his publications.

In [8], Graves-Morris and Saff proved an analogue of the direct part of Gonchar's Theorem for simultaneous approximation with the aid of the concept of polewise independence of a system of functions (for the definition, see [8]). They also established upper bounds for the convergence rates corresponding to (3) and (4). The Graves-Morris/Saff Theorem was refined and complemented in [2, Theorem 4.4] by weakening the assumption of polewise independence, improving the upper bound given in [8] for the rate (3), and giving the exact one for (4). Until now, results of inverse type for row sequences of Hermite-Padé approximants are not available.

Our purpose is to obtain an analogue of Gonchar's Theorem for simultaneous Hermite-Padé approximants, characterizing the exact rates of convergence of the $Q_{n,\mathbf{m}}$ and $\mathbf{R}_{n,\mathbf{m}}$.

The underlying idea in inverse-type results is that a polynomial which is the limit of the denominators of the approximants must have as zeros the poles of the function being approximated, provided that the rate of convergence is geometric. However, the actual situation in simultaneous approximation may be rather complicated as the following example shows. Take $\mathbf{f} = (f_1, f_2)$, where

$$f_1 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!}, \tag{5}$$

and $\mathbf{m} = (1,1)$. It is clear that the unit circle is a natural boundary of definition for both functions f_1 and f_2 and thus $z = 2$ cannot be a pole of $\mathbf{f}$ in any system of domains. However, results contained in [2] show that the denominators $Q_{n,\mathbf{m}}$ of the simultaneous Hermite-Padé approximants converge with geometric rate to the polynomial $(z-1/2)(z-2)$.

This kind of examples leads us to introduce the following concept which is actually inspired by the definition of polewise independence in [8].

For each $r > 0$, set $D_r = \{z \in \mathbb{C} : |z| < r\}$, $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$, and $\overline{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.2. *Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$ if for each $s = 1, \ldots, \tau$ there exists at least one polynomial combination of the form*

$$\sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \ldots, d, \tag{6}$$

which is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s and there is no polynomial combination of the form (6) with those properties for s greater than τ. If some component m_k equals zero the corresponding polynomial p_k is taken identically equal to zero.

The great advantage of this definition with respect to that of polewise independence is that we have liberated it from establishing a priori a region where the property should be verified. This turns out to be crucial.

We wish to underline that if some component m_k equals zero, that component places no restriction on Definition 1.1 and does not report any benefit in finding system poles; therefore, without loss of generality we can restrict our attention to multi-indices $\mathbf{m} \in \mathbb{N}^d$, and we will do so in the sequel, except in reference to the convergence of the approximants themselves.

Notice that the definition of system pole strongly depends on the multi-index $\mathbf{m}$ and that a system $\mathbf{f}$ cannot have more than $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting their order. During the proof of Theorem 1.3 below,

carried out in Section 3, we give a procedure for finding in a finite number of steps all the system poles of $\mathbf{f}$ with respect to a multi-index $\mathbf{m}$ under appropriate conditions.

It is easy to see that a system pole may not be a pole of $\mathbf{f}$ or viceversa. For example, let $\mathbf{f}$ be the system given by (5) and $\mathbf{m} = (1, 1)$. The point $z = 2$, which lies beyond the natural boundary of definition of f_1 and f_2 is not a pole; however it is a system pole of $\mathbf{f}$ since $f_1 - f_2$ has a pole at $z = 2$.

On the other hand, take $\mathbf{f} = (f_1, f_2)$ with

$$f_1 = \frac{1}{z-1} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{z-3},$$

and $\mathbf{m} = (1, 1)$. Then the points $z = 1$ and $z = 3$ are poles and system poles of $\mathbf{f}$ but $z = 2$ is only a pole because there is no way of eliminating the pole at $z = 1$ through linear combinations of f_1 and f_2 without eliminating the pole at $z = 2$.

To each system pole ξ of $\mathbf{f}$ with respect to $\mathbf{m}$ we associate several characteristic values. Let τ be the order of ξ as a system pole of $\mathbf{f}$. For each $s = 1, \ldots, \tau$ denote by $r_{\xi,s}(\mathbf{f}, \mathbf{m})$ the largest of all the numbers $R_s(g)$ (the radius of the largest disk containing at most s poles of g), where g is a polynomial combination of type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of order s. Then

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,s} r_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

$$R_{\xi}(\mathbf{f}, \mathbf{m}) = R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{s=1,\ldots,\tau} r_{\xi,s}(\mathbf{f}, \mathbf{m}).$$

Obviously, if $d = 1$ and $(\mathbf{f}, \mathbf{m}) = (f, m)$, system poles and poles in $D_m(f)$ coincide. Also, $R_\xi(\mathbf{f}, \mathbf{m}) = R_m(f)$ for each pole ξ of f in $D_m(f)$.

By $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$ we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$ is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$.

The following theorem constitutes our main result.

Theorem 1.3. *Let* $\mathbf{f}$ *be a system of formal Taylor expansions as in* (1) *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Then, the following two assertions are equivalent.*

a) $R_0(\mathbf{f}) > 0$ *and* $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

b) *The sequence of denominators* $\{Q_{n,\mathbf{m}}\}_{n \geq |\mathbf{m}|}$ *of simultaneous Padé approximations of* $\mathbf{f}$ *is uniquely determined for all sufficiently large* n *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Moreover, if either a) *or* b) *takes place then* $Q_{|\mathbf{m}|} \equiv \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$ *and*

$$\theta = \max\left\{ \frac{|\xi|}{R_\xi(\mathbf{f}, \mathbf{m})} \,:\, \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m}) \right\}. \tag{7}$$

If $d = 1$, $R_{n,m}$ and $Q_{n,m}$ are uniquely determined. Therefore, Theorem 1.3 implies Gonchar's Theorem except for (4) whose analogue will be presented in Section 3.2 to avoid introducing new notation at this stage.

The paper is structured as follows. In Section 2 we continue with the study of incomplete Padé approximants initiated in [2] proving results of inverse type. Section 3 is dedicated to the proof of Theorem 1.3 and the analogue of (4).

2. Incomplete Padé approximants

Let

$$f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \qquad \phi_n \in \mathbb{C}, \tag{8}$$

denote a formal or convergent Taylor expansion about the origin.

Definition 2.1. *Let f denote a formal Taylor expansion as in* (8)*. Fix $m \geq m^* \geq 1$. Let $n \geq m$. We say that the rational function $r_{n,m}$ is an incomplete Padé approximation of type (n, m, m^*) corresponding to f if $r_{n,m}$ is the quotient of any two polynomials p and q that verify*

b.1) $\deg p \leq n - m^*, \quad \deg q \leq m, \quad q \not\equiv 0,$
b.2) $q(z)f(z) - p(z) = Az^{n+1} + \cdots .$

Notice that given $(n, m, m^*), n \geq m \geq m^*$, any of the Padé approximants $R_{n,m^*}, \ldots, R_{n,m}$ can be regarded an incomplete Padé approximation of type (n, m, m^*) of f. From Definition 1.1 and (2) it follows that $R_{n,\mathbf{m},k}, k = 1, \ldots, d$, is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

In the sequel, for each $n \geq m \geq m^*$, we choose one candidate. After canceling out common factors between q and p, we write $r_{n,m} = p_{n,m}/q_{n,m}$, where, additionally, $q_{n,m}$ is normalized to be monic. Suppose that q and p have a common zero at $z = 0$ of order λ_n. From b.1)-b.2) it follows that

b.3) $\deg p_{n,m} \leq n - m^* - \lambda_n, \quad \deg q_{n,m} \leq m - \lambda_n, \quad q_{n,m} \not\equiv 0,$
b.4) $q_{n,m}(z)f(z) - p_{n,m}(z) = Az^{n+1-\lambda_n} + \cdots .$

where A is, in general, a different constant from the one in b.2).

The first difficulty encountered in dealing with inverse-type results is to justify in terms of the data that the formal series corresponds to an analytic element which does not reduce to a polynomial. In our aid comes the next result, which provides such information in terms of whether the zeros of the polynomials $q_{n,m}$ remain away or not from 0 and/or ∞ as n grows. Let

$$\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}, \quad n \geq m, \quad m_n \leq m,$$

denote the collection of zeros of $q_{n,m}$ repeated according to their multiplicity, where $\deg q_{n,m} = m_n$. Put

$$S = \sup_{N \geq m} \inf \{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\}$$

and

$$G = \inf_{N \geq m} \sup \{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\}.$$

Finally, set

$$\tau_n = \min\{n - m^* - \lambda_n - \deg p_{n,m}, m - \lambda_n - m_n\}, \quad n \geq m.$$

From b.3) we know that $\tau_n \geq 0$, $n \geq m$.

Theorem 2.2. *Let f be a formal power series as in* (8)*. Fix $m \geq m^* \geq 1$. The following assertions hold.*

i) *If $|\lambda_n - \lambda_{n-1}| \leq m^* - 1, n \geq n_0$, and $S > 0$ then $R_0(f) > 0$.*
ii) *If $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1, n \geq n_0$, and $G < \infty$ then either f is a polynomial or $R_0(f) < \infty$. If, additionally, there exists a sequence of indices Λ such that* $\deg q_{n,m} \geq 1, n \in \Lambda$, *then $R_0(f) < \infty$.*

Proof. From definition

$$(q_{n,m} f - p_{n,m})(z) = A z^{n+1-\lambda_n} + \cdots, \tag{9}$$

and $q_{n,m}(0) \neq 0$.

We may suppose that $\inf \{|\zeta_{n,k}| : n \geq n_0, m_n \geq 1, 1 \leq k \leq m_n\} > 0$ and $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$, $n \geq n_0$. Normalize $q_{n,m}$ as follows. If $m_n \geq 1$ take

$$q_{n,m}(z) = \prod_{k=1}^{m_n} \left(1 - \frac{z}{\zeta_{n,k}}\right) = a_{n,0} + a_{n,1} z + \cdots + a_{n,m_n} z^{m_n}, \quad a_{n,0} = 1.$$

Otherwise $q_{n,m}(z) \equiv 1 = a_{n,0}$.

Using the Vieta formulas connecting the coefficients of a polynomial and its zeros it follows that there exists $C_1 \geq 1$ such that

$$\sup \{|a_{n,k}| : 0 \leq k \leq m_n,\ n \geq n_0\} \leq C_1 < \infty. \tag{10}$$

The coefficient corresponding to $z^k, k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$ in the left-hand side of (9) equals

$$\phi_k + a_{n,1}\phi_{k-1} + \cdots + a_{n,m_n}\phi_{k-m_n} = 0, \tag{11}$$

since $\deg p_{n,m} \leq n - m^* - \lambda_n$.

If $m_n \geq 1$, (10) and (11) imply that

$$|\phi_k| \leq C_1(|\phi_{k-1}| + \cdots + |\phi_{k-m_n}|).$$

Therefore, for each $k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$ there exists $k' \in \{k-1, \ldots, k-m\}$ ($m_n \leq m$) such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{12}$$

Should $m_n = 0$, for the same values of k, we have $\phi_k = 0$ and (12) is trivially verified. Substituting n by $n-1$, we deduce that for each $k \in \{n - m^* - \lambda_{n-1}, \ldots, n - \lambda_{n-1} - 1\}$ there exists $k' \in \{k-1, \ldots, k-m\}$ such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{13}$$

As $n \geq n_0$, we have

$$n - \lambda_{n-1} \geq n - \lambda_n - m^* + 1$$

and

$$n - \lambda_{n-1} - m^* \leq n - \lambda_n - 1,$$

because $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$. Consequently, the range of values taken by k due to relations (12) and (13) are either contiguous or overlapping for $n \geq n_0$. Since $n - \lambda_n$ tends to ∞ as n goes to ∞, we conclude that for all $n \geq n_0$ there exists $n' \in \{n-1, \ldots, n-m\}$ such that

$$|\phi_n| \leq C_1 m |\phi_{n'}|. \tag{14}$$

Let Λ be a sequence of indices such that

$$\lim_{n \in \Lambda} |\phi_n|^{1/n} = \limsup_{n \to \infty} |\phi_n|^{1/n} = 1/R_0(f).$$

Choose $n \in \Lambda$. Due to (14) there exist indices $n_1 > n_2 > \cdots > n_{r_n}$, $n_{r_n} \leq n_0$, where $r_n \leq n - n_0$, such that

$$|\phi_n| \leq C_1 m |\phi_{n_1}| \leq \cdots \leq (C_1 m)^{r_n} |\phi_{n_{r_n}}|.$$

Consequently,

$$1/R_0(f) = \lim_{n \in \Lambda} |\phi_n|^{1/n} \leq \limsup_{n \to \infty} (C_1 m)^{r_n/n} \leq C_1 m.$$

Therefore, $R_0(f) \geq (C_1 m)^{-1} > 0$, which proves i).

As for ii), assume that $\sup\{|\zeta_{n,k}| : n \geq n_0, m_n \geq 1, 1 \leq k \leq m_n\} < \infty$ and $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1$, $n \geq n_0$. Set $t_n(z) = (z-1)^{\tau_n}$. Define $\tilde{q}_{n,m} = t_n q_{n,m}$ and $\tilde{p}_{n,m} = t_n p_{n,m}$. Normalize $\tilde{q}_{n,m}$ as follows. If $m_n + \tau_n \geq 1$ take

$$\tilde{q}_{n,m}(z) = \prod_{k=1}^{m_n+\tau_n} (z - \zeta_{n,k}) = b_{n,0} z^{m_n+\tau_n} + \cdots + b_{n,m_n+\tau_n-1} z + b_{n,m_n+\tau_n},$$

where $b_{n,0} = 1$. Should $m_n + \tau_n = 0$ we set $\tilde{q}_{n,m} \equiv 1 = b_{n,0}$. Using the Vieta formulas, it follows that there exists $C_2 \geq 1$ such that

$$\sup\{|b_{n,k}| : 0 \leq k \leq m_n,\ n \geq n_0\} \leq C_2 < \infty. \tag{15}$$

The coefficient corresponding to $z^k, k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$, in the left-hand side of (9) equals

$$\phi_{k-m_n-\tau_n} + b_{n,1}\phi_{k-m_n-\tau_n+1} + \cdots + b_{n,m_n+\tau_n}\phi_k = 0, \tag{16}$$

since $\deg \tilde{p}_{n,m} \leq n - m^* - \lambda_n$.

Should $m_n + \tau_n \geq 1$, (15) and (16) imply that

$$|\phi_{k-m_n-\tau_n}| \leq C_2(|\phi_{k-m_n-\tau_n+1}| + \cdots + |\phi_k|),$$

or, what is the same, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$, we have

$$|\phi_k| \leq C_2(|\phi_{k+1}| + \cdots + |\phi_{k+m_n+\tau_n}|).$$

Therefore, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$ there exists $k' \in \{k+1, \ldots, k+m\}$ $(m_n + \tau_n \leq m)$ such that

$$|\phi_{k'}| \geq \frac{|\phi_k|}{C_2 m}. \tag{17}$$

In case that $m_n + \tau_n = 0$ we have $\phi_k = 0$ for the same values of k and (17) is also true.

Using the assumption that $|\lambda_n + m_n + \tau_n - \lambda_{n-1} - m_{n-1} - \tau_{n-1}| \leq m^* - 1$, it is easy to check, similarly to the previous case, that the range of values taken by the parameter k for consecutive values of n are either contiguous or overlapping. Also, $n - \lambda_n - m_n - \tau_n$ tends to ∞ as n goes to ∞. Consequently, from (17) we have that for all $n \geq n_0$ there exists $n' \in \{n+1, \ldots, n+m\}$ such that

$$|\phi_{n'}| \geq \frac{|\phi_n|}{C_2 m} \tag{18}$$

Using (18) we can find an increasing sequence of multi-indices $\{n_s\}_{s\in\mathbb{Z}_+}$, $n_{s+1} \in \{n_s + 1, \ldots, n_s + m\}$ and $n_1 \in \{n_0, \ldots, n_0 + m\}$ such that

$$|\phi_{n_{s+1}}| \geq \frac{|\phi_{n_1}|}{(C_2 m)^s}.$$

Should f be a polynomial there is nothing to prove. Otherwise, changing the value of n_0 if necessary, without loss of generality we can assume that $\phi_{n_1} \neq 0$. Then,

$$\liminf_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}} \geq \frac{1}{\limsup\limits_{s\to\infty} (C_2 m)^{s/n_{s+1}}} \geq \frac{1}{C_2 m},$$

since

$$\limsup_{s\to\infty} \frac{s}{n_{s+1}} \leq \limsup_{s\to\infty} \frac{s}{n_1 + s} = 1.$$

It follows that

$$R_0(f) = \frac{1}{\limsup\limits_{n\to\infty} |\phi_n|^{1/n}} \leq \frac{1}{\liminf\limits_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}}} \leq C_2 m < \infty,$$

as we needed to prove.

Finally, if f is a polynomial, say of degree N, we would have that for all $n \geq N + m$, $f \equiv p_{n,m}/q_{n,m}$ and $q_{n,m} \equiv 1$. Consequently, if there exists Λ such that $\deg q_{n,m} \geq 1, n \in \Lambda$, f cannot be a polynomial and, therefore, only $R_0(f) < \infty$ is possible. □

Lemma 2.3. *A sufficient condition to have* $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$ *and* $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1$ *is that*

$$\min\{m_n + \tau_n,\, m_{n-1} + \tau_{n-1}\} \geq m - m^* + 1.$$

Proof. In fact, for $k = n-1$ and $k = n$, if $m_k + \tau_k \geq m - m^* + 1$ then $0 \leq \lambda_k \leq m^* - 1$ because $\lambda_k + m_k + \tau_k \leq m$ and the first inequality readily follows. On the other hand,

$$|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})|$$

$$= |(m_n + \lambda_n + \tau_n - m + m^* - 1) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1} - m + m^* - 1)|$$

and $0 \leq m_k + \lambda_k + \tau_k - m + m^* - 1 \leq m^* - 1$ for $k = n-1$ and $k = n$. Therefore, the second inequality also holds. □

Applied to Padé approximation ($m^* = m$), Theorem 2.2 and Lemma 2.3 imply that if $\deg Q_{n,m} \geq 1$ and its zeros remain uniformly bounded away from 0 and ∞, for sufficiently large n, then $0 < R_0(f) < \infty$. This result has not been stated elsewhere.

Let us see some consequences of Theorem 2.2 and Lemma 2.3 on the extendability of a formal power series and the location of some of its poles in terms of the behavior of the zeros of the approximants. First we bring your attention to some results from [2].

Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$ we denote the class of all coverings of B by at most a numerable set of disks. Set

$$\sigma(B) = \inf\left\{\sum_{i=1}^{\infty} |U_i| \,:\, \{U_i\} \in \mathcal{U}(B)\right\},$$

where $|U_i|$ stands for the radius of the disk U_i. The quantity $\sigma(B)$ is called the 1-dimensional Hausdorff content of the set B.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a domain $D \subset \mathbb{C}$ and φ another function defined on D. We say that $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in σ-content to the function φ on compact subsets of D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \sigma\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0.$$

We denote this writing $\sigma\text{-}\lim_{n\to\infty} \varphi_n = \varphi$ inside D.

We define the number $R^*_m(f)$ as the radius of the largest disk centered at the origin on compact subsets of which the sequence $\{r_{n,m}\}_{n\geq m}$ converges to f in σ-content. In [2] we gave a formula to produce this number and showed that it depends on the specific sequence of incomplete Padé approximants considered. Set $D^*_m(f) = \{z \in \mathbb{C} : |z| < R^*_m(f)\}$.

Among other direct-type results, we proved that

$$R_{m^*}(f) \leq R^*_m(f) \leq R_m(f), \tag{19}$$

that $R_m^*(f) > 0$ implies $R_0(f) > 0$, and that each pole of the function f in $D_m^*(f)$ attracts, with geometric rate, at least as many zeros of $q_{n,m}$ as its order (see [2, Theorem 3.5]). Therefore, Theorem 2.2 and Lemma 2.3 imply

Corollary 2.4. *Let f be a formal power series as in* (8). *Fix $m \geq m^* \geq 1$. Assume that there exists a polynomial q_m of degree greater than or equal to $m - m^* + 1$, $q_m(0) \neq 0$, such that $\lim_{n\to\infty} q_{n,m} = q_m$. Then $0 < R_0(f) < \infty$ and the zeros of q_m contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

We need a relaxed version of Corollary 2.4 for the proof of Theorem 1.3.

Lemma 2.5. *Let f be a formal power series as in* (8) *that is not a polynomial. Fix $m \geq m^* \geq 1$. Let $r_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition 2.1 and common factors between them are allowed. Assume that there exists a polynomial $\tilde{q}_m$ of degree m, $\tilde{q}_m(0) \neq 0$, such that $\lim_{n\to\infty} \tilde{q}_{n,m} = \tilde{q}_m$. Then $0 < R_0(f) < \infty$ and the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

Proof. Let us show that the assumptions of Lemma 2.3 are verified for the incomplete approximant $r_{n,m}$. Let $r_{n,m} = p_{n,m}/q_{n,m}$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. Since $\tilde{q}_m(0) \neq 0$, then $\tilde{q}_{n,m}(0) \neq 0$, $n \geq n_0$. Thus, $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z = 0$ and $\lambda_n = 0$ for all $n \geq n_0$. As before, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\left\{n - m^* - \deg p_{n,m},\, m - m_n\right\}, \qquad n \geq n_0.$$

Notice that $\tau_n = m - m_n$, $n \geq n_0$, because the polynomials $q_{n,m}$ and $p_{n,m}$ are obtained eliminating possible common factors between $\tilde{q}_{n,m}$ and $\tilde{p}_{n,m}$ and by assumption

$$\min\left\{n - m^* - \deg \tilde{p}_{n,m},\, m - \deg \tilde{q}_{n,m}\right\} = 0, \qquad n \geq n_0.$$

Therefore, we have

$$m_n + \tau_n = m \geq m - m^* + 1, \qquad n \geq n_0,$$

and Lemma 2.3 is applicable.

From Theorem 2.2 we obtain $0 < R_0(f) < \infty$. Now, from the fact that each pole of f in $D_m^*(f)$ attracts as many zeros of $q_{n,m}$ as its order it follows that the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$. □

In case that there exists $R > R_{m^*}(f)$ inside of which f is meromorphic then D_R contains at least $m^* + 1$ poles of f since $D_{m^*}(f)$ is the largest disk where f is meromorphic with at most m^* poles. We can prove the following inverse-type result.

Theorem 2.6. *Fix $m \geq m^* \geq 1$. Let f be a formal power series as in* (8) *that is not a rational function with at most $m^* - 1$ poles. Let $r_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*)*

corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition 2.1 and common factors between them are allowed. Suppose that there exists a polynomial $\tilde{q}_m$, of degree m, $\tilde{q}_m(0) \neq 0$, such that

$$\limsup_{n\to\infty} \|\tilde{q}_{n,m} - \tilde{q}_m\|^{1/n} = \theta < 1. \tag{20}$$

Then, either f has exactly m^* poles in $D_{m^*}(f)$, which are zeros of $\tilde{q}_m$ counting multiplicities, or $R_0(\tilde{q}_m f) > R_{m^*}(f)$.

Proof. From Lemma 2.5 we have $R_0(f) > 0$. So, f is analytic in a neighborhood of $z = 0$. We also know that $R_0(\tilde{q}_m f) \geq R_{m^*}(f)$ since the zeros of $\tilde{q}_m$ contain all the poles that f has in $D_{m^*}(f)$. Assume that $R_0(\tilde{q}_m f) = R_{m^*}(f)$. Let us show that then f has exactly m^* poles in $D_{m^*}(f)$. To the contrary, suppose that f has in $D_{m^*}(f)$ at most $m^* - 1$ poles. Then there exists a polynomial q_{m^*}, with $\deg q_{m^*} < m^*$, such that

$$R_0(q_{m^*} f) = R_{m^*}(f) = R_0(q_{m^*} \tilde{q}_m f).$$

Let

$$q_{m^*}(z)\, \tilde{q}_m(z)\, f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$R_{m^*}(f) = R_0(q_{m^*} \tilde{q}_m f) = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

The n-th Taylor coefficient of $q_{m^*}[\tilde{q}_{n,m} f - \tilde{p}_{n,m}]$ is equal to zero. Therefore, the n-th Taylor coefficients of $q_{m^*} \tilde{q}_m f$ and $q_{m^*} \tilde{q}_m f - q_{m^*} \tilde{q}_{n,m} f + q_{m^*} p_{n,m}$ coincide. Take $0 < r < R_{m^*}(f)$ and recall that $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Hence

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{[q_{m^*} \tilde{q}_m f - q_{m^*} \tilde{q}_{n,m} f + q_{m^*} p_{n,m}](\omega)}{\omega^{n+1}}\, d\omega \\ &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{[\tilde{q}_m - \tilde{q}_{n,m}](\omega)\, q_{m^*}(\omega) f(\omega)}{\omega^{n+1}}\, d\omega. \end{aligned}$$

Making use of (20) it readily follows that

$$\frac{1}{R_{m^*}(f)} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \frac{\theta}{r}.$$

Letting r tend to $R_{m^*}(f)$ we have

$$\frac{1}{R_{m^*}(f)} \leq \frac{\theta}{R_{m^*}(f)}, \qquad \theta < 1,$$

which implies that $R_{m^*}(f) = \infty$. Let us show that this is not possible.

In fact,

$$[q_{m^*}\, \tilde{q}_{n,m}\, f - q_{m^*}\, \tilde{p}_{n,m}]\,(z) = A_n z^{n+1} + \cdots,$$

and $\deg q_{m^*} \tilde{p}_{n,m} \leq n - 1$. It follows that $(q_{m^*} \tilde{p}_{n,m})/\tilde{q}_{n,m} = (q_{m^*} p_{n,m})/q_{n,m}$ is an incomplete Padé approximant of the function $q_{m^*} f$ of type $(n, m, 1)$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. As $\tilde{q}_{n,m}(0) \neq$

0, $n \geq n_0$, the polynomials $q_{m^*}\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z = 0$ and $\lambda_n = 0$ for all $n \geq n_0$. Again, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\{n - 1 - \deg p_{n,m},\, m - m_n\}.$$

Notice that $\tau_n = m - m_n$, $n \geq n_0$, because

$$\min\{n - 1 - \deg q_{m^*}\tilde{p}_{n,m},\, m - \deg \tilde{q}_{n,m}\} = 0, \quad n \geq n_0.$$

Thus, $m_n + \tau_n = m$, $n \geq n_0$. Using Lemma 2.3 (for $m^* = 1$) and Theorem 2.2 we conclude that either $R_0(q_{m^*}f) < \infty$ or $q_{m^*}f$ is a polynomial. However, the latter is not possible by hypotheses. On the other hand, $R_0(q_{m^*}f) < \infty$ contradicts $R_{m^*}(f) = \infty$. As claimed, f has exactly m^* poles in $D_{m^*}(f)$. □

3. Simultaneous approximation

Throughout this section, $\mathbf{f} = (f_1, \ldots, f_d)$ denotes a system of formal power expansions as in (1) and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ is a fixed multi-index. We are concerned with the simultaneous approximation of $\mathbf{f}$ by sequences of vector rational functions defined according to Definition 1.1 taking account of (2). That is, for each $n \in \mathbb{N}, n \geq |\mathbf{m}|$, let $(R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d})$ be a Hermite-Padé approximation of type $(n, \mathbf{m})$ corresponding to $\mathbf{f}$.

As we mentioned earlier, $R_{n,\mathbf{m},k}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k, $k = 1, \ldots, d$. Thus, from (19) we have

$$D_{m_k}(f_k) \subset D^*_{|\mathbf{m}|}(f_k) \subset D_{|\mathbf{m}|}(f_k), \quad k = 1, \ldots, d.$$

Definition 3.1. *A vector $\mathbf{f} = (f_1, \ldots, f_d)$ of formal power expansions is said to be algebraically independent with respect to $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ if there do not exist polynomials $p_1, \ldots, p_d$, at least one of which is non-null, such that*

c.1) $\deg p_k \leq m_k - 1$, $k = 1, \ldots, d$,
c.2) $\sum_{k=1}^{d} p_k f_k$ *is a polynomial.*

In particular, algebraic independence implies that for each $k = 1, \ldots, d$, f_k is not a rational function with at most $m_k - 1$ poles. Notice that algebraic independence may be verified solely in terms of the coefficients of the formal Taylor expansions defining the system $\mathbf{f}$.

Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we consider the associated system $\overline{\mathbf{f}}$ of formal power expansions

$$\overline{\mathbf{f}} = (f_1, \ldots, z^{m_1-1}f_1, f_2, \ldots, z^{m_d-1}f_d) = (\bar{f}_1, \ldots, \bar{f}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\overline{\mathbf{m}}$ given by $\overline{\mathbf{m}} = (1, 1, \ldots, 1)$ with $|\overline{\mathbf{m}}| = |\mathbf{m}|$. The systems $\mathbf{f}$ and $\overline{\mathbf{f}}$ share most properties. In particular, poles of $\mathbf{f}$ and $\overline{\mathbf{f}}$ coincide and $R_m(\mathbf{f}) = R_m(\overline{\mathbf{f}})$, $m \in \mathbb{Z}_+$.

From the definition it readily follows that $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$ if and only if there do not exist constants $c_k, k =$

$1, \ldots, |\mathbf{m}|$, not all zero, such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k \bar{f}_k$$

is a polynomial. That is, $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$ if and only if $\overline{\mathbf{f}}$ is algebraically independent with respect to $\overline{\mathbf{m}}$. By the same token, the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ are the same as the system poles of $\overline{\mathbf{f}}$ with respect to $\overline{\mathbf{m}}$.

Finally, it is very easy to check that, for all $n \geq |\mathbf{m}|$, the equations that define the common denominator $Q_{n,\mathbf{m}}$ for $(\mathbf{f}, \mathbf{m})$ are the same as those defining $Q_{n,\overline{\mathbf{m}}}$ for $(\overline{\mathbf{f}}, \overline{\mathbf{m}})$ and, consequently, both classes of polynomials coincide.

Lemma 3.2. *Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of formal Taylor expansions as in* (1) *and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Suppose that for all $n \geq n_0$ the polynomial $Q_{n,\mathbf{m}}$ is unique and $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. Then, the system $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$.*

Proof. Because of what said just before the statement of Lemma 3.2, we can assume without loss of generality that $\mathbf{m} = (1, 1, \ldots, 1)$ and $d = |\mathbf{m}|$. We argue by contradiction. Suppose that there exist constants c_k, $k = 1, \ldots, d$, not all zero, such that $\sum_{k=1}^{d} c_k f_k$ is a polynomial. Should $d = 1$, $Q_{n,\mathbf{m}} \equiv 1$ for all n sufficiently large and $\deg Q_{n,\mathbf{m}} < 1 = |\mathbf{m}|$. If $d > 1$, without loss of generality, we can assume that $c_1 \neq 0$. Then

$$f_1 = p - \sum_{k=2}^{d} c_k f_k,$$

where p is a polynomial, say of degree N.

On the other hand, for each $n \geq d - 1$, there exist polynomials $Q_n, P_{n,k}$, $k = 2, \ldots, d$, such that

- $\deg P_{n,k} \leq n - 1$, $k = 2, \ldots, d$, $\quad \deg Q_n \leq d - 1$, $\quad Q_n \not\equiv 0$,
- $Q_n(z)\, f_k(z) - P_{n,k}(z) = A_k z^{n+1} + \cdots, \quad k = 2, \ldots, d.$

Therefore,

$$Q_n(z)\left(p(z) - \sum_{k=2}^{d} c_k f_k(z)\right) - \left(Q_n(z)\, p(z) - \sum_{k=2}^{d} c_k P_{n,k}(z)\right) = A z^{n+1} + \ldots$$

and, for $n \geq d + N$, the polynomial $P_{n,1} = Q_n\, p - \sum_{k=2}^{d} c_k P_{n,k}$ verifies $\deg P_{n,1} \leq n - 1$. Thus, for all n sufficiently large, the polynomials $P_{n,k}$, $k = 1, \ldots, d$, satisfy Definition 1.1 with respect to $\mathbf{f}$ and $\mathbf{m}$. Naturally, Q_n gives rise to a polynomial $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < d = |\mathbf{m}|$ against our assumption on $Q_{n,\mathbf{m}}$. $\square$

Set

$$\mathbf{D}^*_{\mathbf{m}}(\mathbf{f}) = \left(D^*_{|\mathbf{m}|}(f_1), \ldots, D^*_{|\mathbf{m}|}(f_d)\right).$$

The following corollaries are straightforward consequences of Corollary 2.4 and Theorem 2.6, respectively, together with the fact that, for each $k = 1, \dots, d$, $R_{n,\mathbf{m},k} = P_{n,\mathbf{m},k}/Q_{n,\mathbf{m}}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Corollary 3.3. *Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of formal Taylor expansions as in* (1) *and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that $\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$. Then $R_0(\mathbf{f}) > 0$, the zeros of $Q_{|\mathbf{m}|}$ contain all the poles that $\mathbf{f}$ has in $\mathbf{D}^*_{\mathbf{m}}(\mathbf{f})$, and $R_0(f_k) < \infty$ for each $k = 1, \dots, d$.*

Corollary 3.4. *Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of formal Taylor expansions as in* (1) *and fix a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Then, for each $k = 1, \dots, d$, either f_k has exactly m_k poles in $D_{m_k}(f_k)$ or $R_0(Q_{|\mathbf{m}|} f_k) > R_{m_k}(f_k)$.

3.1. **Proof of Theorem 1.3.** Let us prove first that b) implies a). From Lemma 3.2 it follows that $\mathbf{f}$ is algebraically independent with respect to $\mathbf{m}$ and, in turn, from Corollary 3.3 we know that $R_0(\mathbf{f}) > 0$. So, it is enough to prove that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and without loss of generality we can assume that $\mathbf{m} = (1, 1, \dots, 1)$.

We divide the proof into two parts. First, we collect a set of $|\mathbf{m}|$ candidates to be system poles of $\mathbf{f}$ and prove that they are the zeros of $Q_{|\mathbf{m}|}$. We also prove that any system pole of $\mathbf{f}$ must be among these candidates. In the second part we prove that all these points previously collected are actually system poles of $\mathbf{f}$.

In the disk $D_0(\mathbf{f})$ there cannot be system poles of $\mathbf{f}$ since all the functions f_k are analytic. Now, for each $k = 1, \dots, d$, by Corollaries 3.4 and 3.3, either the disk $D_1(f_k)$ contains exactly one pole of f_k, and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} f_k) > R_1(f_k)$. Therefore, $D_0(\mathbf{f}) \neq \mathbb{C}$ and $Q_{|\mathbf{m}|}$ contains as zeros all the poles of f_k on the boundary of $D_0(f_k)$ counting their order for $k = 1, \dots, d = |\mathbf{m}|$. Moreover, the functions f_k cannot have on the boundary of $D_0(f_k)$ singularities other than poles.

According to this, the poles of $\mathbf{f}$ on the boundary of $D_0(\mathbf{f})$ are all zeros of $Q_{|\mathbf{m}|}$ counting multiplicities and the boundary contains no other singularity except poles. Let us call them candidate system poles of $\mathbf{f}$ and denote them by $a_1, \dots, a_{n_1}$ taking account of their order. Obviously, any system pole of $\mathbf{f}$ on the boundary of $D_0(\mathbf{f})$ must be one of the candidates since no linear combination of the functions in $\mathbf{f}$ can produce poles at any other point.

Since $\deg Q_{|\mathbf{m}|} = |\mathbf{m}|$ we have $n_1 \leq |\mathbf{m}|$. Should $n_1 = |\mathbf{m}|$ we have found all the candidates we were looking for. Let us assume that $n_1 < |\mathbf{m}|$. We

can find coefficients $c_1, \dots, c_{|\mathbf{m}|}$ such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k f_k$$

is analytic in a neighborhood of $\overline{D_0(\mathbf{f})}$. Finding the coefficients c_k reduces to solving a linear homogeneous system of n_1 equations with $|\mathbf{m}|$ unknowns. In fact, if $z = a$ is a candidate system pole of $\mathbf{f}$ with multiplicity τ we obtain τ equations choosing the coefficients c_k so that

$$\int_{|\omega-a|=\delta} (\omega - a)^i \left(\sum_{k=1}^{|\mathbf{m}|} c_k f_k(\omega) \right) d\omega = 0, \qquad i = 0, \dots, \tau - 1. \tag{21}$$

where δ is sufficiently small. We do the same with each distinct candidate on the boundary of $D_0(\mathbf{f})$. The linear homogeneous system of equations so obtained has at least $|\mathbf{m}| - n_1$ linearly independent solutions which we denote by $\mathbf{c}_j^1$, $j = 1, \dots, |\mathbf{m}| - n_1^*$, $n_1^* \le n_1$.

Set

$$\mathbf{c}_j^1 = (c_{j,1}^1, \dots, c_{j,|\mathbf{m}|}^1), \quad j = 1, \dots, |\mathbf{m}| - n_1^*.$$

Construct the $(|\mathbf{m}| - n_1^*) \times |\mathbf{m}|$ dimensional matrix

$$C^1 = \begin{pmatrix} \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*}^1 \end{pmatrix}.$$

Define the system $\mathbf{g}_1$ of $|\mathbf{m}| - n_1^*$ functions by means of

$$\mathbf{g}_1^t = C^1 \mathbf{f}^t = (g_{1,1}, \dots, g_{1,|\mathbf{m}|-n_1^*})^t,$$

where $(\cdot)^t$ means taking transpose. We have

$$g_{1,j} = \sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 f_k, \qquad j = 1, \dots, |\mathbf{m}| - n_1^*.$$

As the rows of C^1 are non-null, none of the functions $g_{1,j}$ are polynomials because of the algebraic independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \dots, 1)$.

Consider the region

$$D_0(\mathbf{g}_1) = \bigcap_{j=1}^{|\mathbf{m}|-n_1^*} D_0(g_{1,j}).$$

Obviously, by construction, $D_0(\mathbf{f})$ is strictly included in $D_0(\mathbf{g}_1)$ and there cannot be system poles of $\mathbf{f}$ in $D_0(\mathbf{g}_1) \setminus \overline{D_0(\mathbf{f})}$.

It is easy to see that

$$\sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 \frac{P_{n,\mathbf{m},k}}{Q_{n,\mathbf{m}}}$$

is an $(n, |\mathbf{m}|, 1)$ incomplete Padé approximant of $g_{1,j}$. Using Theorem 2.6 with $m^* = 1$, for each $j = 1, \ldots, |\mathbf{m}| - n_1^*$, either the disk $D_1(g_{1,j})$ contains exactly one pole of $g_{1,j}$, and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} g_{1,j}) > R_1(g_{1,j})$. In particular, $D_0(\mathbf{g}_1) \neq \mathbb{C}$ and all the singularities of $\mathbf{g}_1$ on the boundary of $D_0(\mathbf{g}_1)$ are poles which are zeros of $Q_{|\mathbf{m}|}$ counting their order. They constitute the next layer of candidate system poles of $\mathbf{f}$ (now, it is possible that some candidates are not poles of $\mathbf{f}$ since the functions f_k intervene in the linear combination as we saw in example (5)). All the system poles of $\mathbf{f}$ on the boundary of $D_0(\mathbf{g}_1)$ must necessarily be poles of $\mathbf{g}_1$ for the same reason as in the preceding case.

Let us denote these new candidates by $a_{n_1+1}, \ldots, a_{n_1+n_2}$. Of course $n_1 + n_2 \leq |\mathbf{m}|$. Should $n_1 + n_2 = |\mathbf{m}|$, we are done. Otherwise, $n_2 < |\mathbf{m}| - n_1 \leq |\mathbf{m}| - n_1^*$ and we can repeat the process. In order to eliminate the n_2 poles we have $|\mathbf{m}| - n_1^*$ functions which are analytic on $D_0(\mathbf{g}_1)$ and meromorphic on a neighborhood of $\overline{D_0(\mathbf{g}_1)}$. The corresponding homogeneous linear system of equations, similar to (21), has at least $|\mathbf{m}| - n_1^* - n_2$ linearly independent solutions $\mathbf{c}_j^2, j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, $n_2^* \leq n_2$. Set

$$\mathbf{c}_j^2 = (c_{j,1}^2, \ldots, c_{j,|\mathbf{m}|-n_1^*}^2), \quad j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*.$$

Construct the $(|\mathbf{m}| - n_1^* - n_2^*) \times |\mathbf{m}| - n_1^*$ dimensional matrix

$$C^2 = \begin{pmatrix} \mathbf{c}_1^2 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*-n_2^*}^2 \end{pmatrix}.$$

Define the system $\mathbf{g}_2$ of $|\mathbf{m}| - n_1^* - n_2^*$ functions by means of

$$\mathbf{g}_2^t = C^2 \mathbf{g}_1^t = C^2 C^1 \mathbf{f}^t = (g_{2,1}, \ldots, g_{2,|\mathbf{m}|-n_1^*-n_2^*})^t.$$

The rows of C^2C^1 are of the form $\mathbf{c}_j^2 C^1, j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, where C^1 has rank $|\mathbf{m}| - n_1^*$ and the vectors $\mathbf{c}_k^2$ are linearly independent. Therefore, the rows of C^2C^1 are linearly independent; in particular, they are non-null. Consequently, the components of $\mathbf{g}_2$ are not polynomials because of the algebraic independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \ldots, 1)$. Thus, we can apply again Theorem 2.6. The proof is completed using finite induction.

Notice that the numbers $n_1, n_2, \ldots$ which so arise are greater than or equal to 1 and on each iteration their sum is less than or equal to $|\mathbf{m}|$. Therefore, in a finite number of steps their sum must equal $|\mathbf{m}|$. Consequently, the number of candidate system poles of $\mathbf{f}$ in some disk, counting their multiplicities, is exactly equal to $|\mathbf{m}|$ and they are precisely the zeros of $Q_{|\mathbf{m}|}$ as we wanted to prove.

Now, suppose that there exists a candidate system pole of $\mathbf{f}$ that is not such or being a system pole has order smaller than the multiplicity of the corresponding zero of $Q_{|\mathbf{m}|}$. Then, for some $\alpha \in \mathbb{Z}_+$, we have

$$\mathbf{g}_\alpha = (g_{\alpha,1}, \ldots, g_{\alpha,\nu}), \quad \nu = |m| - n_0^* - n_1^* - \cdots - n_\alpha^*,$$

with $0 \leq n_j^* \leq n_j$, $j = 0, 1, \ldots, \alpha$, $n_0 = 0$, and $\mu = |\mathbf{m}| - n_0 - n_1 - \cdots - n_\alpha > 0$ such that there exists a point a on the boundary of the region

$$D_0(\mathbf{g}_\alpha) = \bigcap_{j=1}^{\nu} D_0(g_{\alpha,j})$$

that is a pole of order τ of g_{α,j_0} for some $j_0 \in \{1, \ldots, \nu\}$ but is not a system pole of $\mathbf{f}$ or is one of order less than τ. Let $a_{n_\alpha+1}, \ldots, a_{n_{\alpha+1}}$, $n_{\alpha+1} \geq \tau$, be the singularities of the functions $g_{\alpha,j}$ on the boundary of $D_0(\mathbf{g}_\alpha)$ counting multiplicities. We distinguish two cases. First, suppose that $n_1 + \cdots + n_{\alpha+1} = |\mathbf{m}|$; then $n_{\alpha+1} = \mu \leq \nu$. All the functions $g_{\alpha,j}$ admit meromorphic extension to a neighborhood of $\overline{D_0(\mathbf{g}_\alpha)}$. We pose the problem of finding coefficients $c_1, \ldots, c_\nu$ such that

$$\sum_{j=1}^{\nu} c_j g_{\alpha,j}$$

is analytic on a neighborhood of $\overline{D_0(\mathbf{g}_\alpha)}$. The problem consists in solving a linear homogeneous system with μ equations and ν unknowns similar to (21) but, due to the fact that the point a is not a system pole of $\mathbf{f}$ or it is one of order less than τ, one of the equations may be written as a linear combination of the others and we have at most $\mu - 1$ equations, with $\mu - 1 < \nu$. So, a non-trivial solution necessarily exists which defines a function g analytic on a neighborhood of $\overline{D_0(\mathbf{g}_\alpha)}$ by means of

$$g = \sum_{j=1}^{\nu} c_j g_{\alpha,j} = \sum_{k=1}^{|\mathbf{m}|} d_k f_k, \quad d_k \in \mathbb{C}, \quad k = 1, \ldots, |\mathbf{m}|.$$

Following the same argument used in the process carried out to find the candidate system poles of $\mathbf{f}$, we deduce that g is not a polynomial. Now,

$$\sum_{k=1}^{|\mathbf{m}|} d_k \frac{P_{n,\mathbf{m},k}}{Q_{n,\mathbf{m}}}$$

is an $(n, |\mathbf{m}|, 1)$ incomplete Padé approximant of g. Using Theorem 2.6 with $m^* = 1$, either the disk $D_1(g)$ contains exactly one pole of g, and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} g) > R_1(g)$. But both alternatives are impossible since all the zeros of $Q_{|\mathbf{m}|}$ belong to $\overline{D_0(\mathbf{g}_\alpha)}$. So, we have reached a contradiction.

In case that $n_1 + \cdots + n_{\alpha+1} < |\mathbf{m}|$ we are in the middle of the process described above and now, when solving the corresponding system of equations to eliminate the $n_{\alpha+1}$ poles, we obtain $n_{\alpha+1}^* < n_{\alpha+1}$ since, again, one of the equations is redundant. This implies that, in the last step, say β, when $n_1 + \cdots + n_\beta = |\mathbf{m}|$ we have $|\mathbf{m}| - n_0 - n_1 - \cdots - n_\beta = \mu < \nu = |\mathbf{m}| - n_0^* - n_1^* - \cdots - n_\beta^*$ reaching the same contradiction as before. We have proved a posteriori that $n_j^* = n_j$, $j = 1, 2, \ldots$

Thus, the proof of the inverse-type result is complete. Also, we have that $Q_{|\mathbf{m}|} \equiv \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$.

Let us prove now that a) implies b). Except for some details related to the numbers $R_\xi(\mathbf{f},\mathbf{m})$, where ξ is a system pole of $\mathbf{f}$, the arguments are similar to those employed in [8]. In spite of this, for completeness, we give the entire proof.

For each $n \geq |\mathbf{m}|$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=1}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \qquad q_{n,\mathbf{m}}(z) = \sum_{k=1}^{|\mathbf{m}|} \lambda_{n,k} z^k. \tag{22}$$

Due to this normalization, the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

Let ξ be a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$. Consider a polynomial combination g_1 of type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a simple pole at $z = \xi$ and verifies that $R_1(g_1) = R_{\xi,1}(\mathbf{f},\mathbf{m})$ $(= r_{\xi,1}(\mathbf{f},\mathbf{m}))$. Then, we have

$$g_1 = \sum_{k=1}^{|\mathbf{m}|} p_{k,1} f_k, \quad \deg p_{k,1} < m_k, \quad k = 1, \ldots, |\mathbf{m}|,$$

and

$$q_{n,\mathbf{m}}(z)\, h_1(z) - (z-\xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z)\, P_{n,\mathbf{m},k}(z) = A z^{n+1} + \ldots,$$

where $h_1(z) = (z-\xi)\, g_1(z)$. Hence, the function

$$\frac{q_{n,\mathbf{m}}(z)\, h_1(z)}{z^{n+1}} - \frac{z-\xi}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z)\, P_{n,\mathbf{m},k}(z)$$

is analytic on $D_1(g_1)$. Take $0 < r < R_1(g_1)$ and set $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Using Cauchy's formula, we obtain

$$q_{n,\mathbf{m}}(z) h_1(z) - (z-\xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_1(\omega)}{\omega - z}\, d\omega,$$

for all z with $|z| < r$, since $\deg \sum_{k=1}^{|\mathbf{m}|} p_{k,1} P_{n,\mathbf{m},k} < n$. In particular, taking $z = \xi$ in the above formula, we arrive at

$$q_{n,\mathbf{m}}(\xi)\, h_1(\xi) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\xi^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_1(\omega)}{\omega - \xi}\, d\omega. \tag{23}$$

Straightforward calculations lead to

$$\limsup_{n\to\infty} |h_1(\xi) q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\xi|}{r}.$$

Using that $h_1(\xi) \neq 0$ and making r tend to $R_1(g_1)$ we obtain

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\xi|}{R_{\xi,1}(\mathbf{f},\mathbf{m})} < 1.$$

Now, we employ induction. Suppose that

$$\limsup_{n\to\infty}\left|q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n}\le\frac{|\xi|}{R_{\xi,j+1}(\mathbf{f},\mathbf{m})},\quad j=0,1,\dots,s-2 \tag{24}$$

(recall that $R_{\xi,j+1}(\mathbf{f},\mathbf{m})=\min_{k=1,\dots,j+1}r_{\xi,k}(\mathbf{f},\mathbf{m})$), with $s\le\tau$, and let us prove that formula (24) holds for $j=s-1$.

Consider a polynomial combination g_s of the type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order s at $z=\xi$ and verifies that $R_s(g_s)=r_{\xi,s}(\mathbf{f},\mathbf{m})$. Then, we have

$$g_s=\sum_{k=1}^{|\mathbf{m}|}p_{k,s}f_k,\quad \deg p_{k,s}<m_k,\quad k=1,\dots,|\mathbf{m}|.$$

Set $h_s(z)=(z-\xi)^s g_s(z)$. Reasoning as in the previous case, the function

$$\frac{q_{n,\mathbf{m}}(z)\,h_s(z)}{z^{n+1}(z-\xi)^{s-1}}-\frac{z-\xi}{z^{n+1}}\sum_{k=1}^{|\mathbf{m}|}p_{k,s}(z)\,P_{n,\mathbf{m},k}(z)$$

is analytic on $D_s(g_s)\setminus\{\xi\}$. Put $P_s=\sum_{k=1}^{|\mathbf{m}|}p_{k,s}P_{n,\mathbf{m},k}$. Fix an arbitrary compact set $K\subset(D_s(g_s)\setminus\{\xi\})$. Take $\delta>0$ sufficiently small and $0<r<R_s(g_s)$ with $K\subset D_r$. Using Cauchy's integral formula and the residue theorem, for all $z\in K$, we have

$$\frac{q_{n,\mathbf{m}}(z)h_s(z)}{(z-\xi)^{s-1}}-(z-\xi)\,P_s(z)=I_n(z)-J_n(z), \tag{25}$$

where

$$I_n(z)=\frac{1}{2\pi i}\int_{\Gamma_r}\frac{z^{n+1}}{\omega^{n+1}}\frac{q_{n,\mathbf{m}}(\omega)\,h_s(\omega)}{(\omega-\xi)^{s-1}(\omega-z)}\,d\omega$$

and

$$J_n(z)=\frac{1}{2\pi i}\int_{|\omega-\xi|=\delta}\frac{z^{n+1}}{\omega^{n+1}}\frac{q_{n,\mathbf{m}}(\omega)\,h_s(\omega)}{(\omega-\xi)^{s-1}(\omega-z)}\,d\omega.$$

We have used in (25) that $\deg P_s<n$. The first integral I_n is estimated as in (23) to obtain

$$\limsup_{n\to\infty}\|I_n(z)\|_K^{1/n}\le\frac{\|z\|_K}{R_s(g_s)}=\frac{\|z\|_K}{r_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{26}$$

As for J_n, write

$$q_{n,\mathbf{m}}(\omega)=\sum_{j=0}^{|\mathbf{m}|}\frac{q^{(j)}_{n,\mathbf{m}}(\xi)}{j!}(\omega-\xi)^j.$$

Then

$$J_n(z)=\sum_{j=0}^{s-2}\frac{1}{2\pi i}\int_{|\omega-\xi|=\delta}\frac{z^{n+1}}{\omega^{n+1}}\frac{q^{(j)}_{n,\mathbf{m}}(\xi)}{j!(\omega-z)}\frac{h_s(\omega)}{(\omega-\xi)^{s-1-j}}\,d\omega. \tag{27}$$

Using the inductive hypothesis (24), estimating the integral in (27), and making ε tend to zero, we obtain

$$\limsup_{n\to\infty} \|J_n(z)\|_K^{1/n} \leq \frac{\|z\|_K}{|\xi|}\frac{|\xi|}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})} = \frac{\|z\|_K}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})},$$

which, together with (26) and (25), gives

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}}(z)h_s(z) - (z-\xi)^s\, P_s(z)\|_K^{1/n} \leq \frac{\|z\|_K}{R_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{28}$$

As the function inside the norm in (28) is analytic in $D_s(g_s)$, inequality (28) also holds for any compact set $K \subset D_s(g_s)$. Besides, we can differentiate $s-1$ times that function and the inequality still holds true by virtue of Cauchy's integral formula. So, taking $z=\xi$ in (28) for the differentiated version, we obtain

$$\limsup_{n\to\infty} \left|\left(q_{n,\mathbf{m}}h_s\right)^{(s-1)}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})}.$$

Using the Leibnitz formula for higher derivatives of a product of two functions and the induction hypothesis (24), we arrive at

$$\limsup_{n\to\infty} \left|q_{n,\mathbf{m}}^{(s-1)}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})},$$

since $h_s(\xi) \neq 0$. This completes the induction.

Let $\xi_1,\dots,\xi_p$ be the distinct system poles of $\mathbf{f}$ and let τ_i be the order of ξ_i as a system pole, $i=1,\dots,p$. By assumption, $\tau_1+\cdots+\tau_p = |\mathbf{m}|$. We have proved that, for $i=1,\dots,p$ and $j=0,1,\dots,\tau_i-1$,

$$\limsup_{n\to\infty} \left|q_{n,\mathbf{m}}^{(j)}(\xi_i)\right|^{1/n} \leq \frac{|\xi_i|}{R_{\xi_i,j+1}(\mathbf{f},\mathbf{m})} \leq \frac{|\xi_i|}{R_{\xi_i}(\mathbf{f},\mathbf{m})}. \tag{29}$$

Recall that $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$ is the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$. Denote by $L_{i,j}$, $i=1,\dots,p$; $j=0,1,\dots\tau_i-1$, the fundamental interpolating polynomials at the zeros of $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$; that is, for each $i=1,\dots,p$ and $j=0,1,\dots\tau_i-1$, $\deg L_{i,j} \leq |\mathbf{m}|-1$ and

$$L_{i,j}^{(\nu)}(b_\kappa) = \delta_{i\kappa}\delta_{j\nu}, \quad \kappa=1,\dots,p, \quad \nu=0,1,\dots,\tau_i-1.$$

Then

$$q_{n,\mathbf{m}}(z) = \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) + \sum_{i=1}^{p}\sum_{j=0}^{\tau_i-1} q_{n,\mathbf{m}}^{(j)}(\xi_i)\, L_{i,j}(z). \tag{30}$$

From (29) and (30) it follows that

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|_K^{1/n} \leq \theta < 1,$$

for any compact $K \subset \mathbb{C}$, where

$$\theta = \max\left\{\frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})}\,:\, \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\right\}. \tag{31}$$

As all norms in finite dimensional spaces are equivalent, we obtain

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} \le \theta < 1. \tag{32}$$

Now, necessarily we have

$$\liminf_{n\to\infty} |\lambda_{n,|\mathbf{m}|}| > 0, \tag{33}$$

since if there exists a subsequence $\Lambda \subset \mathbb{N}$ such that $\lim_{n\in\Lambda} \lambda_{n,|\mathbf{m}|} = 0$, then from (32) we have $\lim_{n\in\Lambda} \|q_{n,\mathbf{m}}\| = 0$, contradicting (22).

As $q_{n,\mathbf{m}} = \lambda_{n,|\mathbf{m}|}Q_{n,\mathbf{m}}$, we have proved

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} \le \theta < 1, \tag{34}$$

where θ is given by (31). In particular, for $n \ge n_0$, $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. The difference of any two non-collinear solutions Q_1 and Q_2 of Definition 1.1 with the same degree and equal leading coefficient produces a new solution of smaller degree but we have proved that any solution must have degree $|\mathbf{m}|$ for all sufficiently large n. Hence, the polynomial $Q_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n. With this we have concluded the proof of the direct result.

Let us prove that the upper bound in (34) actually gives the exact rate of convergence to obtain (7). To the contrary, suppose that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} = \theta' < \theta. \tag{35}$$

Let ζ be a system pole of $\mathbf{f}$ such that

$$\frac{|\zeta|}{R_\zeta(\mathbf{f},\mathbf{m})} = \theta = \max\left\{\frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})} \,:\, \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\right\}.$$

Naturally, if there is inequality in (35) then $R_\zeta(\mathbf{f},\mathbf{m}) < \infty$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k = 1,\dots,d, \tag{36}$$

that is analytic on a neighborhood of $\overline{D}_{|\zeta|}$ except for a pole of order s at $z=\zeta$ with $R_s(g) = R_\zeta(\mathbf{f},\mathbf{m})$. On the boundary of $D_s(g)$ the function g must have a singularity which is not a system pole. In fact, if all the singularities were of this type we could find a different polynomial combination g_1 of type (36) for which $R_s(g_1) > R_s(g) = R_\zeta(\mathbf{f},\mathbf{m})$ against our definition of $R_\zeta(\mathbf{f},\mathbf{m})$. For short, put $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) = Q_{|\mathbf{m}|}$. Consequently, the function $Q_{|\mathbf{m}|}\, g$ can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_\zeta(\mathbf{f},\mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_\zeta(\mathbf{f},\mathbf{m}). \tag{37}$$

On the other hand, by virtue of (36), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)\, g(z) - \sum_{k=1}^{d} p_k(z)\, P_{n,\mathbf{m},k}(z) = B_n z^{n+1} + \dots$$

and this function is analytic at least in $D_{|\zeta|}$ with a zero of multiplicity at least $n+1$ at $z=0$. Taking $r < |\zeta|$, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{H_n(\omega)}{\omega^{n+1}}\, d\omega = 0.$$

Set $P_n = \sum_{k=1}^{d} p_k P_{n,\mathbf{m},k}$. Clearly, $Q_{|\mathbf{m}|}\, g \equiv (Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}})\, g + P_n + H_n$ and, since $\deg P_n \leq n-1$, we arrive at

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{Q_{|\mathbf{m}|}(\omega)\, g(\omega)}{\omega^{n+1}}\, d\omega = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{\left[Q_{|\mathbf{m}|}(\omega) - Q_{n,\mathbf{m}}(\omega)\right] g(\omega)}{\omega^{n+1}} d\omega.$$

Taking (37) and (35) into consideration, estimating the integral, and letting r tend to $|\zeta|$, it follows that

$$\frac{1}{R_\zeta(\mathbf{f},\mathbf{m})} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \frac{\theta'}{|\zeta|} < \frac{\theta}{|\zeta|} = \frac{1}{R_\zeta(\mathbf{f},\mathbf{m})},$$

which is absurd. We have completed the proof of Theorem 1.3. □

3.2. **Convergence of the Hermite-Padé approximants.** The following result is in some sense the analogue of the formula displayed just after (58) in [4] written in different terms.

Theorem 3.5. *Assume that either* a) *or* b) *in Theorem 1.3 takes place. If ξ is a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$, then*

$$\max_{j=0,\dots,\overline{s}} \limsup_{n\to\infty} \left|Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} = \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s} = 0,1,\dots,\tau-1. \tag{38}$$

Proof. Let ξ be as indicated. From (29) and (33) we have

$$\max_{j=0,\dots,\overline{s}} \limsup_{n\to\infty} \left|Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s} = 0,1,\dots,\tau-1.$$

Assume that there is strict inequality for some $\overline{s} \in \{0,\dots,\tau-1\}$ and fix $\overline{s}$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k = 1,\dots,d,$$

that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order s ($\leq \overline{s}+1$) at $z=\xi$ with $R_s(g) = R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$. As before, on the boundary of $D_s(g)$ the function g must have a singularity which is not a system pole. Set $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) = Q_{|\mathbf{m}|}$. Consequently, the function $Q_{|\mathbf{m}|}\, g$ can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m}). \tag{39}$$

On the other hand, by virtue of (36), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)\, g(z) - \sum_{k=1}^{d} p_k(z)\, P_{n,\mathbf{m},k}(z) = B_n z^{n+1} + \dots$$

and this function is analytic in $D_s(g) \setminus \{\xi\}$. Take r smaller than but sufficiently close to $R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$ and $\delta > 0$ sufficiently small. Let $\Gamma_{\delta,r}$ be the positively oriented curve determined by $\gamma_\delta = \{\omega : |\omega - \xi| = \delta\}$ and Γ_r. We have

$$\frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{H_n(\omega)}{\omega^{n+1}}\, d\omega = 0.$$

Set $P_n = \sum_{k=1}^d p_k P_{n,\mathbf{m},k}$ and $h(\omega) = (\omega-\xi)^s g(\omega)$. Obviously,

$$Q_{|\mathbf{m}|}\, g \equiv (Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}})\, g + P_n + H_n$$

and, since $\deg P_n \le n-1$, we obtain

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{Q_{|\mathbf{m}|}(\omega) g(\omega)}{\omega^{n+1}}\, d\omega = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega) h(\omega)}{(\omega-\xi)^s \omega^{n+1}} d\omega$$

$$= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega) h(\omega)}{(\omega-\xi)^s \omega^{n+1}} d\omega - \sum_{\nu=0}^{|\mathbf{m}|} \frac{1}{2\pi i}\int_{\gamma_\delta} \frac{[Q^{(\nu)}_{|\mathbf{m}|} - Q^{(\nu)}_{n,\mathbf{m}}](\xi) h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}} d\omega$$

$$= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega) h(\omega)}{(\omega-\xi)^s \omega^{n+1}} d\omega - \sum_{\nu=0}^{s-1} \frac{1}{2\pi i}\int_{\gamma_\delta} \frac{Q^{(\nu)}_{n,\mathbf{m}}(\xi) h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}} d\omega.$$

Estimating these integrals, using (7) and the temporary assumption that

$$\max_{j=0,\dots,\overline{s}} \limsup_{n\to\infty} \left| Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} = \frac{|\xi|}{\kappa} < \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})},$$

we obtain

$$\limsup_{n\to\infty} |c_n|^{1/n} \le \max\left\{\frac{1}{\kappa}, \frac{\theta}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}\right\} < \frac{1}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})},$$

which contradicts (39). Hence, (38) takes place. $\square$

Now, we are ready to give the analogue of (4) for simultaneous approximation. We need to introduce some notation. Fix $k \in \{1,\dots,d\}$. Let $D_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ be the largest disk centered at $z=0$ in which all the poles of f_k are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of f_k does not exceed their order as system poles, and f_k has no other singularity. By $R_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ we denote the radius of this disk. Let $\xi_1,\dots,\xi_N$ be the poles

of f_k in $D_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. For each $j=1,\ldots,N$, let $\widetilde{\tau}_j$ be the order of ξ_j as a pole of f_k and τ_j its order as a system pole. By assumption $\widetilde{\tau}_j \le \tau_j$. Set

$$R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) = \min\left\{R_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}), \min_{j=1,\ldots,N} R_{\xi_j,\widetilde{\tau}_j}(\mathbf{f},\mathbf{m})\right\},$$

and let $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ be the disk centered at $z=0$ with this radius.

Recall that $\sigma(B)$ stands for the 1-dimensional Hausdorff content of the set B. We say that a compact set $K\subset\mathbb{C}$ is σ-regular if for each $z_0\in K$ and for each $\delta>0$, it holds that $\sigma\{z\in K : |z-z_0|<\delta\}>0$.

Theorem 3.6. *Let $\mathbf{f}$ be a system of formal Taylor expansions as in* (1) *and fix a multi-index $\mathbf{m}\in\mathbb{Z}^d_+\setminus\{\mathbf{0}\}$. Suppose that either* a) *or* b) *in Theorem 1.3 takes place. Then,*

$$\limsup_{n\to\infty}\|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \le \frac{\|z\|_K}{R^*_{m_k}(f_k)}, \quad k=1,\ldots,d, \tag{40}$$

where K is any compact subset of $D^*_{|\mathbf{m}|}(f_k)\setminus\mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$. If, additionally, K is σ-regular, then we have equality in (40)*. Moreover,*

$$R^*_{m_k}(f_k) = R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}), \qquad k=1,\ldots,d.$$

Proof. Let us fix $k\in\{1,\ldots,d\}$ and maintain the notation introduced above. Let K be a compact subset contained in $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})\setminus\mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$. Take r smaller than but sufficiently close to $R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$, and $\delta>0$ sufficiently small so that K is in the region bounded by Γ_r and the circles $\{z : |z-\xi_j| = \delta\}$, $j=1,\ldots,N$. Let $\Gamma_{\delta,r}$ be the curve with positive orientation determined by Γ_r and those circles. On account of Definition 1.1, using Cauchy's integral formula we have

$$(Q_{n,\mathbf{m}}f_k - P_{n,\mathbf{m},k})(z) = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{z^{n+1}}{\omega^{n+1}}\frac{(Q_{n,\mathbf{m}}f_k)(\omega)}{\omega - z}d\omega$$

Since $\lim_n Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$, using (38) and standard arguments we obtain

$$\limsup_{n\to\infty}\|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \le \frac{\|z\|_K}{R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})}. \tag{41}$$

This last relation implies that $\sigma\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$ inside $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. Since $R^*_{m_k}(f_k)$ is the largest disk inside of which such convergence takes place it readily follows that $R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) \le R^*_{m_k}(f_k)$. Should $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ contain on its boundary some singularity which is not a system pole then necessarily $R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) = R^*_{m_k}(f_k)$ because σ-convergence implies that all singularities inside must be zeros of $Q_{|\mathbf{m}|}$ but the zeros of this polynomial are all system poles as we proved in Theorem 1.3. Assume that $R^*_{m_k}(f_k) > R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. Then, we have $R^*_{m_k}(f_k) > \min_{j=1,\ldots,N} R_{\xi_j,\widetilde{\tau}_j}(\mathbf{f},\mathbf{m})$. From the proof of [2, Theorem 3.6] we know that for each pole ξ of order $\widetilde{\tau}$ of f_k inside $D^*_{m_k}(f_k)$

$$\limsup_{n\to\infty}\left|Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} \le \frac{|\xi|}{R^*_{m_k}(f_k)}, \quad j=0,1,\ldots,\widetilde{\tau}-1.$$

This contradicts (38). Consequently $R^*_{m_k}(f_k) = R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ as claimed.

Due to (41), we have also proved (40). In order to show that this formula is exact for σ-regular compact subsets one must argue as in the corresponding part of the proof of [2, Theorem 4.4]. □

As compared with [2, Theorem 4.4], Theorem 3.6 offers weaker assumptions and a characterization of the values $R^*_{m_k}(f_k)$ in terms of the analytic properties of the functions in the system instead of the coefficients of their Taylor expansion. An open question is to obtain an analogous characterization when the assumptions of Theorem 3.6 do not take place.

It would be interesting to study inverse problems for row sequences of Hermite-Padé approximation when only the limit behavior of some of the zeros of the polynomials $Q_{n,\mathbf{m}}$ is known, in the spirit of the conjectures proposed by A.A. Gonchar in [4].

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